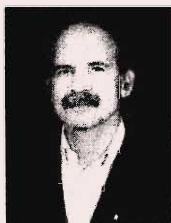


James Clerk Maxwell and his Equations

B N Dwivedi



B N Dwivedi does research in solar physics and teaches physics in Banaras Hindu University. He has over twenty two years of teaching experience and broad experience in solar research with involvement in almost all the major solar space experiments, including Skylab, Yohkoh, SOHO, and TRACE. The Max-Planck-Institut für Aeronomie recently awarded him the 'Gold Pin' in recognition of his outstanding contribution to the SOHO/Summer experiment.

This article presents a brief account of life and work of James Clerk Maxwell and his equations.

James Clerk Maxwell

James Clerk Maxwell was a physicists' physicist, the prime author of the modern theory of colour vision, the principal creator of statistical thermodynamics, and above all the author of the classical electromagnetic theory, with its identification of light and electromagnetic waves. Maxwell's electromagnetic theory is acknowledged as one of the outstanding achievements of nineteenth century physics. If you wake up a physicist in the middle of the night and say 'Maxwell', I am sure he will say 'electromagnetic theory'. Einstein described the change brought about by Maxwell in the conception of physical reality as "the most profound and the most fruitful that physics has experienced since the time of Newton". Maxwell's description of reality is represented in his double system of partial differential equations in which the electric and magnetic fields appear as dependent variables. Since Maxwell's time physical reality has been thought of as represented by continuous fields governed by partial differential equations. The advent of quantum mechanics, rather than the theory of relativity, has produced a situation in which we have to deal with amounts of energy concentrated in small spaces. This is in direct contradiction to Maxwell's equations and to Newtonian mechanics. The quantities that appear in the laws of quantum mechanics make no claim to describe physical reality as such, they describe the probabilities of the appearance of a particular physical reality. Whatever be the difficulties, Maxwell's theory will always remain a step of outstanding importance in the development of physical ideas.

Maxwell has been described by his biographers as one who has enriched the inheritance left by Newton and has consolidated

Keywords

Life and work of Maxwell, Maxwell field equations, Hertz's experiment.

the work of Faraday, one who impelled the mind of Cambridge to a fresh course of real investigation, a man of profound and original genius, who clearly earned his place in human memory.

Maxwell as a Student

Maxwell was born in Edinburgh on 13 June 1831 but spent his early childhood (1831-41) at Glenlair in Scotland. He showed remarkable curiosity at an early age, asking questions such as “What’s the go of that?”, “What does it do?” and if he was not satisfied with the answer he would ask “But what is the ‘particular go’ of it?”. Besides asking questions, he was very fond of making things such as baskets and seals covered with strange devices. His mother died in 1839 at the early age of 48 from abdominal cancer, to which he was to succumb forty years later when he was exactly the same age. A dull and uninspired tutor was engaged for him and he claimed that James was slow at learning. He attended the school at Edinburgh Academy during 1841-47. While in school, at the age of 14, he published his first scientific paper ‘On the description of oval curves and those having a plurality of foci’. This fascination for geometry and for mechanical models continued through his whole career, and was of great help in his subsequent research. He entered Edinburgh University in 1847 and during this period (1847-50) he published two more scientific papers. He went on to Cambridge in 1850 where his exceptional powers began to be recognized. His mathematics teacher was William Hopkins, the famous ‘Wrangler maker’ of the period whose students included G G Stokes, William Thomson, Arthur Cayley, P G Tait and E J Routh. Besides reading with Hopkins, he also attended Stokes’ lectures. Of Maxwell Hopkins is reported to have said that it seemed impossible for him to think wrongly on any physical subject, but that in analysis he was far more deficient. This shrewd assessment was borne out later by several formulae advanced by Maxwell where the results have proved to be correct but the mathematical arguments leading to them were found faulty. In 1854 he graduated as a second Wrangler and was also the second Smith’s Prizeman. It is interesting that the paper set by Stokes



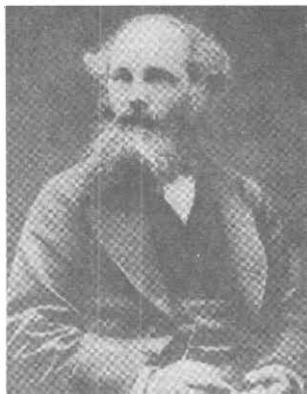
(photo credit: Prof. Sir John C. Gunn)

James Clerk Maxwell (1831-79) signed himself dp/dt in accordance with the thermodynamic relation $dp/dt = JCM$.

James as a student at Cambridge.

(photo credit: Prof. Sir John C. Gunn)





(photo credit: Nature 282, 1979)

James in his later years.

in the examination contained Stokes' theorem, published for the first time. This fundamental theorem (connecting a line with a surface integral) was to prove of vital importance in the development of Maxwell's electric theory. He published in 1856 the essay on 'On the stability of the motion of Saturn's rings' which won him the Adams Prize.

Maxwell as a Teacher

Maxwell was appointed to the Chair of Natural Philosophy at Marischal College, Aberdeen in 1856. He held this appointment for three sessions until the professorship was suppressed. He suffered a great personal loss when his father, with whom he had a close relationship, died in 1856. Maxwell must have felt then

the need for his own family life and in June 1858 he married Katherine Mary Dewar, daughter of the Principal of Marischal College. The union (like Faraday's) was childless and described as a 'married life ... of unexampled devotion'. In 1860 Maxwell applied for the Chair of Natural Philosophy at Edinburgh but was unsuccessful (his old school friend and rival in the Tripos, P G Tait, was appointed). But in the same year, he was appointed to the Chair of Natural Philosophy at King's College, London. Maxwell was twenty-nine years of age. He held this post until 1865. The official historian of King's College has left the impression that in 1865 Maxwell was asked to resign his Chair on account of his alleged lack of teaching ability and poor class management. After Maxwell retired to Glenlair, he continued to visit

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M 30

Glenlair
Dunblair
1865 Nov 9

Dear Thomson

When I last wrote I had not been at St Andrews. I went last week, and have gone in for the Principalship. If you can certify my having been industrious &c since 1856, or if you can tell me what scientific men are conversant with better or if you can use any influence yourself in my favour pray do. 6 Professors out of 9 have memorialized the Lords. & Home Sec. for me together with Principal Tulloch. the V. Chancellor. Of the other 3, one Prof Sharp, is a candidate and one, Prof Bell does not approve of memorials at all and is neutral, I have written to Sabine Sir Stokes and Grove.

Yours truly
J Clerk Maxwell

Letter to Lord Kelvin

London every spring, and served as an external examiner for the Mathematical Tripos at Cambridge. It is significant to note that Maxwell did most of his work on electricity and magnetism during the period of his retirement (1865-70). On 9 November 1868, Maxwell wrote to Lord Kelvin praying to certify his capability or to use any influence when he unsuccessfully tried for the principalship of the United College of the University of St. Andrews.

In 1870, the descendants of Henry Cavendish endowed the Cavendish Laboratory in Cambridge and created the Cavendish Chair.

In 1871, Maxwell was approached regarding the new Chair but he showed no enthusiasm for the position. In his reply to the Reverend E W Blore on 15th February 1871 he said "I had no intention of applying for the post when I got your letter, and I have none now unless I come to see that I can do some good by it". But then he continued in his letter asking, "Who appoints him and is the appointment for life or during pleasure?" Obviously Maxwell wanted to ascertain if the appointment is for life before he allowed his name to go forward. He was duly elected the first Cavendish Professor of Physics. He held this position until he died of cancer on 5 November 1879, a few months after Einstein was born. Maxwell cared deeply for his wife, who had indifferent health for many years and who surprisingly survived him. Even as he lay dying he gave orders for her comfort. Maxwell had very few students, but they were of the highest calibre, and included W D Niven, Ambrose Fleming, R T Glazebrook, J H Poynting and Arthur Schuster. And it is both inspiring and instructive to note the kind of training students would receive in the hands of Maxwell: "I never", he told Sir Arthur Schuster, "try to dissuade a man from trying an experiment; if he does not find out what he is looking for he may find something else."

Maxwell, in his short life of forty-eight years, published just over a hundred scientific papers. Excluding the geometrical juvenilia,

some papers on mechanics and the Smith's Prize essay already referred to, Maxwell's papers fall into three groups : 1. colour vision and optics; 2. molecular theory (including heat and thermodynamics); 3. electricity and electromagnetic theory. Maxwell's prediction that light consists of electromagnetic waves was a consequence of his work on electromagnetic theory and not on optics as such.

Electromagnetic Theory

At the time Maxwell published his first paper in 1856 on Faraday's lines of force, the mathematical work of Laplace and Poisson had greatly contributed to the understanding of the distribution of electric charge on bodies of various shapes. This led to the study of mechanical forces exerted by the charges on bodies and thereby to the formulation of Coulomb's law – electrified bodies attract or repel each other. Likewise in magnetism, André Ampère had been studying the forces between current-carrying conductors. These developments viewed the subject from the standpoint of action-at-a-distance. As a result, the space outside an electric charge or a magnetic dipole was not supposed to differ from an empty space. Faraday's discovery of electromagnetic induction, however, showed that space occupied by lines of force was not the same as empty space. For instance, a circuit rotated in space through which no magnetic lines of force pass will not register an electric current. Maxwell's objective obviously was to formulate Faraday's idea into a mathematical form. It is, therefore, important to emphasize that Maxwell had an intuition that action-at-a-distance, as was implied by Coulomb's law, for instance, was not true in physics. Having outlined Faraday's notion of lines of force and their representation by iron filings, he was dissatisfied with the action-at-a-distance theories of electricity and magnetism. He went on to introduce a description in which the magnetic dipole, or the electric charge, or the electric current disturbs only the medium in its immediate neighbourhood, and that action is then transmitted from point to point, much as in the case of a wire which we pull to actuate a bell. The tension in the wire propagates from point



to point until finally, after a small but finite time, it arrives at the other end. Thus action will always depend on what happens in the medium and from this Maxwell visualized that disturbances in the electromagnetic field travel with the speed of light.

While mathematising the work of Michael Faraday, Maxwell perceived that if a magnetic field was wrapped round a current, a current could equally be wrapped round a magnetic field. The relation between them was reciprocal and could be expressed mathematically by 'curl' (the word curl was first coined by Maxwell himself). When the two fields co-existed in space, the reaction between them could be expressed as the curl of a curl and this he simplified down to the well-known wave equation, the velocity of propagation of the wave being the reciprocal of the geometric mean of an electric and a magnetic constant. He advanced the subject as a consequence of his recognition that a changing electric field constitutes a current which he named as 'displacement current' (Maxwell's real discovery!) and is accompanied by a magnetic field; and that light consists of electromagnetic waves propagating with a speed of 3×10^8 m/s.

Maxwell's theory predicts the extremely important idea that electrical disturbances are transverse waves of electric and magnetic fields, the velocity of propagation of which depends upon quantities which can be determined purely by electrical measurements and should be equal to the velocity of light. We are most familiar with the waves that we see on the surface of water. Another type of wave is produced by vibrating the end of a long rope or wire. All wave motions involve exchanges between two kinds of energy. In a water wave, for instance, the water moves up and its weight makes it fall back again. The exchange is between energy of motion and the gravitational energy associated with weight. The disturbance at one place affects the adjoining water and so gives the appearance of a travelling wave. In a vibrating wire or a rope, energy is transferred between motion and tension. In the electromagnetic wave, however, the exchange is between electrical energy and magnetic energy. Because the wave motion depends on the exchanges between

electrical and magnetic fields, Maxwell deduced that the speed is simply the ratio of force expressed in magnetic units to that expressed in electrical units, and this value is precisely the same as the velocity of light.

Maxwell was a prophetic genius who lived in the nineteenth century but his physical ideas belonged to the twentieth century. Maxwell's genius, like that of Newton and Faraday, was timeless. Maxwell predicted the electromagnetic theory of radiation and thus set in train a series of events that still profoundly affect our daily lives. Maxwell's theory was not very enthusiastically received because of its obscure presentation and the earlier success of the action-at-a-distance theory. This obscurity was in Hertz's mind when he said that Maxwell's theory was Maxwell's equations. Maxwell also discussed the problem of the gravitational field in his famous paper 'On the Dynamical Theory of the Electromagnetic Field' in 1864 and thus began the long and as yet unsuccessful attempt to derive a unified field theory of matter. Maxwell's description of reality, already referred to, is represented in his double system of partial differential equations and let us discuss them in some detail.

Maxwell's Equations

To discuss this may we remind you of these well-known equations, not in the notation in which Maxwell set them down in his famous paper, but in the notation which is familiar to us today. In SI units these are:

$$\nabla \times \mathbf{E} = \rho/\epsilon \quad \text{Gauss law} \quad (1)$$

$$\nabla \times \vec{\mathbf{E}} = - \frac{\partial \mathbf{B}}{\partial t} \quad \text{Faraday law} \quad (2)$$

$$\nabla \times \mathbf{B} = 0 \quad \text{No magnetic charges} \quad (3)$$

and (for the moment)

$$\nabla \times \vec{\mathbf{B}} = \mu \vec{\mathbf{J}} \quad \text{Ampère law} \quad (4)$$

This incomplete form of the equations summarizes what was

known at the time. Equation (1) states that lines of force can be used to describe the electric field and that these lines of force always begin on positive charges and end on negative charges, as was demonstrated by Faraday. Considering the electric field further, (2) states that there exists an electric potential in a static field so that, as long as only static fields are considered, the energy of a moving particle is conserved. The extra term in (2) represents the law of induction. These differential equations can be shown to be completely equivalent to the integral form of the laws, namely, Coulomb's law, and Faraday's law of induction, as they were then known. Equation (3) expresses the fact that there are also lines of force for the magnetic field, but these do not begin or end anywhere because there are no free magnetic poles. Finally (4) explains that a magnetic field is created near a current, i.e., it is equivalent to the Biot-Savart law.

Box 1.

The charge and current distributions give both \mathbf{E} and \mathbf{B} and the Maxwell equation is just a relation between them. It is easiest to see this using the magnetic vector potential \mathbf{A} (which is the electromagnetic momentum per unit charge). Take curl of

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

$$\begin{aligned} \text{to give } \nabla \times \mathbf{E} &= \text{curl}(-\nabla V) - \text{curl}\left(\frac{\partial \mathbf{A}}{\partial t}\right) \\ &= -\text{curl}\left(\frac{\partial \mathbf{A}}{\partial t}\right). \end{aligned}$$

Since these linear operators commute

$$\text{curl}\left(\frac{\partial \mathbf{A}}{\partial t}\right) = \frac{\partial}{\partial t} \text{curl} \mathbf{A} = \left(\frac{\partial \mathbf{B}}{\partial t}\right).$$

$$\text{Hence the equation } \text{curl} \mathbf{E} = -\left(\frac{\partial \mathbf{B}}{\partial t}\right)$$

is nothing more than

$$-\text{curl}\left(\frac{\partial \mathbf{A}}{\partial t}\right) = -\frac{\partial}{\partial t} (\text{curl} \mathbf{A}).$$

Therefore, we cannot claim that the right hand side generates the left hand side. One can make similar

comments about $\epsilon_0 \left(\frac{\partial \mathbf{E}}{\partial t}\right)$.

Now, if we had access to all the knowledge about electromagnetic field that was available at the time of Maxwell, and if one had formed the view that action over a distance was not the basis of the phenomenon, but that there should be local laws which must be expressed in terms of differential equations rather than as integrals, one would have arrived at (1) to (4). One of the ways in which we teach our students today is to point out that these equations by themselves are not consistent if we assume them to remain valid even in circumstances in which the charges and currents change with time. Taking the divergence of (4) and remembering that divergence of curl of a vector vanishes (physically it means that the curl \mathbf{B} follows the lines of force for which the divergence is zero), we get

$$\nabla \cdot \mathbf{J} = 0, \quad (5)$$

which means that currents involved flow in closed loops. However, we know that $\text{div } \mathbf{J}$ should not vanish in general because if it did, the charge conservation (any flow of charge must come from some supply) equation

$$\nabla \cdot \mathbf{J} = \frac{\partial \rho}{\partial t} = 0 \quad (6)$$

implies that $\partial \rho / \partial t = 0$ hence the charge at each point in space is unchangeable. This is clearly false, since we can change ρ at any point in space by placing a charged body at the point. Thus the Ampère law cannot be a complete statement about the properties of curl \mathbf{B} . It is easy to see that we can correct this inconsistency by adding the extra term $\varepsilon(\partial \mathbf{E} / \partial t)$ to the current in (4), so yielding the correct equation

$$\nabla \times \mathbf{B} = \mu \mathbf{J} + \mu \varepsilon \frac{\partial \mathbf{E}}{\partial t}. \quad (7)$$

Then from (1) and (7), we have the continuity (6). (It is annoying how few people realize that $\varepsilon(\partial \mathbf{E} / \partial t)$ is not only needed for conservation of charge but also for electromagnetic waves!).



We do not necessarily regard this argument as a proof because this is probably not the only way of reconciling the inconsistent equations. One could have modified the equations in other ways to make them consistent; in fact, if we had stuck to the law of action at a distance then we would just have said, 'the derivation without the new term is not valid when the current is varying – one must modify (4) in such a case'. So this is not meant as a rigorous proof but is the argument which makes the form of Maxwell's equations most plausible to the beginner. There is no evidence in Maxwell's papers that this was the path by which he arrived at his result or these arguments played any part in his reasoning. Maxwell arrived at the extra term by using a picture which we do not accept today. If we put an electric field on a capacitor in which there is a medium of very high dielectric constant, then most of the electric induction \mathbf{D} is actually contributed by the separation of charges in the dielectric from one side to the other. It is then natural to expect that the motion of these charges would be accompanied by a current – the displacement current. But we would not postulate any such motion of charges in the vacuum. However, Maxwell, who in fact called \mathbf{D} the electric displacement, had just such a picture in mind.

The need to add the 'displacement current' to the Ampère law was seen by Maxwell as necessary in order that the magnetic field should satisfy the laws of vector analysis when $\text{div } \mathbf{J}$ is not zero. We often give the impression that the famous Maxwell equation was deduced from the Ampère law, but this is not strictly true. Maxwell gave a brilliant, but intuitive, solution to the purely mathematical consideration that the divergence of the curl of any vector must vanish. One may note Maxwell writing to H R Droop on December 28, 1861 stated: "I am trying to form an exact mathematical expression for all that is known about electromagnetism without the aid of hypothesis".

The beginners find it difficult to appreciate the physical meaning of displacement current in free space, where ρ and \mathbf{J} are zero in which case there is no need to modify the Ampère law. It is,



therefore, important to emphasize that the crucial thing Maxwell did was to introduce the vacuum displacement current $\epsilon_0(\partial\mathbf{E}/\partial t)$ which leads to the wave equation (It is of interest to note that one also needs the $-\partial\mathbf{B}/\partial t$ term in $\text{curl } \mathbf{E} = -\partial\mathbf{B}/\partial t$ in order to get the wave equation.). From the pedagogical viewpoint, the student would clearly see that without the vacuum displacement current, one could not form the wave equation in free space. Taking the curl of

$$\text{curl } \mathbf{E} = -\partial\mathbf{B}/\partial t,$$

we obtain

$$\nabla^2\mathbf{E} = \mu_0\epsilon_0 \frac{\partial^2\mathbf{E}}{\partial t^2}. \quad (8)$$

Similarly, taking the curl of

$$\text{curl } \mathbf{B} = \mu_0\epsilon_0 \frac{\partial\mathbf{E}}{\partial t}$$

we obtain

$$\nabla^2\mathbf{B} = \mu_0\epsilon_0 \frac{\partial^2\mathbf{B}}{\partial t^2}, \quad (9)$$

In his letter to Faraday on 19 October 1861, Maxwell said “We have now strong reason to believe, whether my theory is a fact or not, that the luminiferous and electromagnetic medium are one”. Equations (8) and (9) establish the possibility of electromagnetic waves in free space. The speed of these waves is given by $v = 1/\sqrt{\mu_0\epsilon_0} = 3 \times 10^8 \text{ m/s} = \text{speed of light}$, suggestive of the electromagnetic nature of light. This is one of the great unifications of physics. The moment Maxwell conceived the idea of the displacement current, a new era started in the history of mankind. And when relativity came, Newton’s equations were found wanting but not Maxwell’s – they needed no relativistic correction. Events of similar importance have not occurred often, though Newton’s Principia and Einstein’s first paper on relativity would surely qualify.



ON PHYSICAL LINES OF FORCE.

The velocity of light in air, as determined by M. Fizeau*, is 70,843 leagues per second (25 leagues to a degree) which gives

$$V = 314,858,000,000 \text{ millimetres}$$

$$= 195,647 \text{ miles per second} \dots\dots\dots (137).$$

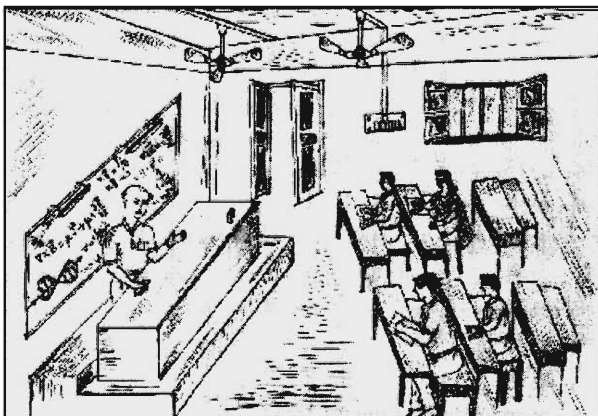
The velocity of transverse undulations in our hypothetical medium, calculated from the electro-magnetic experiments of MM. Kohlrausch and Weber, agrees so exactly with the velocity of light calculated from the optical experiments of M. Fizeau, that we can scarcely avoid the inference that *light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena.*

Experimental Verification

The experimental verification of Maxwell's theory was provided by Rudolf Heinrich Hertz. The results of his experiments were the subject of two papers he published in 1888 entitled 'On Electromagnetic Waves in Air and their Reflection' and 'On Electric Radiation'. Encouraged to study electricity by von Helmholtz of the University of Berlin, Hertz set out to see if electric disturbance travels with infinite speed as Weber thought, or if it is wave-like, as Maxwell predicted. In his experiment Hertz devised a spark-gap generator and used it as a transmitter in which a high voltage produced by an induction coil caused a spark at the gap. As the surge of current due to the spark oscillated between the two plates, radio waves of about 5 m wavelength were generated. Hertz detected the radio waves by means of a metal circle with a very narrow gap as a receiver. A spark set off in the transmitter would produce a spark in the receiver loop. He then set up a fairly large zinc sheet at the other end of his laboratory. By moving his receiving loop toward and away from the zinc sheet, he found that the

Extract from Maxwell's 1862 paper in the Philosophical Magazine in which he first formulated the electromagnetic theory of light.

(Credit: Dr Anita Mohan)



Suggested Reading

- [1] L Campbell and W Garnett, *The Life of James Clerk Maxwell*, MacMillan, London, 1884.
- [2] W D Niven (ed.), *The Scientific Papers of James Clerk Maxwell*, Vol. I, Cambridge University Press, 1890.
- [3] H Hertz, *Electrical Waves* (D E Jones, Trans.), Dover, New York, 1962.
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- [5] C Domb, *James Clerk-Maxwell 100 years later*, *Nature*, Vol.282, pp. 235-239, 1979.
- [6] B N Dwivedi, *A Recourse to Maxwell's Equations*, *Physics Education*, Vol.10, pp.79-82, 1993.

intensity of the received spark varied, thereby confirming the presence of standing waves. His experiments showed that electric and magnetic fields were waves, as Maxwell had predicted. Hertz also used parabolic cylinders to reflect these waves. He was amazed to find that the electric waves passed through wooden doors, that they could be reflected like sunlight and that they were polarized. Thus he firmly established Faraday's view of lines of force and Maxwell's view of waves. The finite velocity of propagation of electromagnetic action dealt a death blow to the 'action-at-a-distance' theories and led to the final vindication of 'action through a medium'. It was now up to other engineers and scientists to discover the vast implications of Maxwell's theory – implications which Maxwell himself might have never realized. Maxwell's revelations have stood the test of time. We have only to sit in our living room and look at pictures of the planet Mars or Jupiter or Saturn transmitted by electromagnetic waves over millions of kilometres through a widely varying medium to become convinced of the genius of this man.

Tribute to Maxwell

Maxwell received no public honours, and was quietly buried in the Parton Churchyard, in Glenlair, Scotland. And perhaps no better tribute to Maxwell could be made than to re-examine the foundations of classical electricity in terms of the physics of today and to state them and teach them in a self-consistent manner.

*Let me dream my dream till morning; let my
mind run slow and clear,
Free from all the world's distraction, feeling that
the Dead are near.
Let me wake, and see my duty lie before me
straight and plain.
Let me rise refreshed, and ready to begin my
work again.*

James Clerk Maxwell

Address for Correspondence
B N Dwivedi
Department of Applied Physics,
Institute of Technology
Banaras Hindu University,
Varanasi 221005, India.
Email :
bholadwivedi@yahoo.com